Flat Plate at an Angle of Attack

It is appropriate to add some comments on the simplest of the Joukowski airfoil solutions, namely planar potential flow past an infinitely thin flat plate at an angle of attack, α. As demonstrated earlier this Joukowski airfoil solution is generated in the limit $R/a \rightarrow 1$, $\beta = 0$ in which the chord of the plate, $c = 4R$. Applying the Kutta condition at the trailing edge and assuming that the flow negotiates the leading edge, the solution yields a lift coefficient, $C_L = 2\pi \sin\alpha$ (with $\Gamma = -4\pi U R \sin\alpha$). Figures 1 and 2 show the z-plane and the ζ-plane for this simple Joukowski airfoil case.

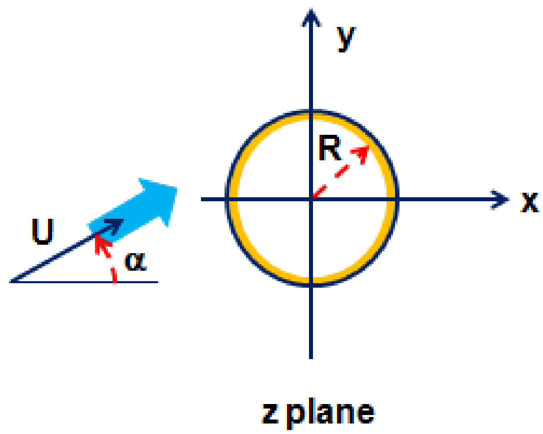

Figure 1: z-plane for the flow around a flat plate at an angle of attack.

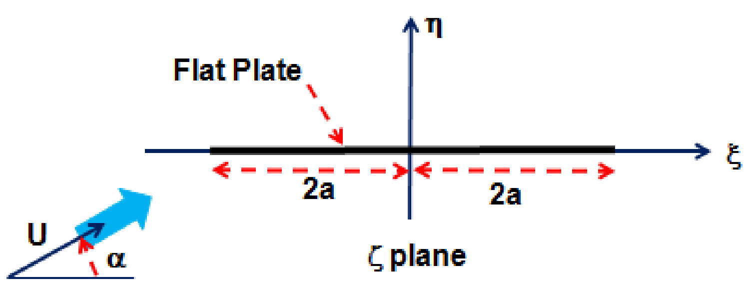

Figure 2: ζ-plane for the flow around a flat plate at an angle of attack.

The surface of the flat plate is given parametrically by

$$\xi = 2R\cos\theta \quad \text{and} \quad \eta = 0 \tag{Bgee1}$$

where $\theta = 0$ is the trailing edge and $\theta = \pi$ is the leading edge. The velocities in the ζ-plane are given by

$$u_\xi = U\left\{\frac{\sin(\theta - \alpha) + \sin\alpha}{\sin\theta}\right\} \quad \text{and} \quad u_\eta = 0 \tag{Bgee2}$$

or

$$u_\xi = U\left[\cos\alpha + \sin\alpha\left\{\frac{(1-\cos\theta)}{(1+\cos\theta)}\right\}^{\frac{1}{2}}\right] = U\left[\cos\alpha + \sin\alpha\left\{\frac{2R-\xi}{2R+\xi}\right\}^{\frac{1}{2}}\right] \tag{Bgee3}$$

and therefore the pressure, p, on the surface of the plate is

$$p = p_\infty + \frac{\rho U^2}{2} - \frac{\rho U^2}{2}\left[\cos\alpha + \sin\alpha\left\{\frac{2R-\xi}{2R+\xi}\right\}^{\frac{1}{2}}\right]^2 \qquad \text{(Bgee4)}$$

To many students one of the subtle and curious aspects of this solution arises when the pressure difference across the infinitely thin foil that results from equation (Bgee4) is integrated to find the force on the foil (per unit span). The result is an upward force equal to $-\rho\Gamma U\cos\alpha$ or $4\pi\rho U^2 R\sin\alpha\cos\alpha$ in a direction perpendicular to the flat plate. This is *not* the same as the expected lift force $-\rho\Gamma U$ in a direction perpendicular to the oncoming stream. The reason for this discrepancy is that there is an additional force, called the *leading edge suction* force that acts parallel to the plate at the leading edge and is equal in magnitude to $\rho\Gamma U\sin\alpha$. When this leading edge suction force is combined with the pressure difference force, the resultant is indeed the lift force perpendicular to the oncoming stream equal in magnitude to $\rho\Gamma U$. Thus the leading edge suction force resolves the apparent discrepancy. When the more general solution for a Joukowski foil of finite thickness and rounded leading edge is examined the specific source of this leading edge suction force can be traced to the detailed pressure distribution around that leading edge. When that is altered by the occurrence in a real viscous flow of separation from the leading edge, that leading edge suction force is lost and the performance of the airfoil is altered.

Real viscous flows cannot negotiate a sharp leading edge as the above potential flow around an infinitely thin flat plate has been permitted to do. In practice at any finite angle of attack, such a foil would experience separation of the flow at the leading edge resulting in a separated region or wake on the upper suction surface and much less lift than the potential flow would predict. The foil would in fact be "stalled". To avoid this at least at small angles of attack (typically less than about 12^o) the leading edge is rounded and the flow remains attached to the suction surface until much further aft; the resulting lift is then close to that of the potential flow prediction. Much care with the design is needed to achieve this end and details such as the effect of small irregularities in the surface of the leading edge can have substantial effects. However, when the angle of attack exceeds some angle of the order of 12^o no design can prevent separation and stall. In summary an effective airfoil design is one with a rounded leading edge that minimizes separation and a sharp trailing edge that encourages smooth separation. Further comments on airfoil performance and design are included in later sections.

[Postscript. It is instructive to recognize that the above potential flow solution for a flat plate at an angle of attack could also have been obtained by the vortex sheet method described earlier. The solution is initiated by assuming that a vortex sheet strength of $\gamma(\xi)$ over the interval $-2R < \xi < 2R$ was necessary to produce the required velocity discontinuity across the infinitely thin foil. The complex potential induced by this vortex sheet follows from equation (Bgeb10) and is

$$f(\zeta) = \int_{-2R}^{+2R} -\frac{i\gamma(\nu)d\nu}{2\pi}\ln(\zeta-\nu) \qquad \text{(Bgee5)}$$

where ν is a dummy ξ variable. Thus the velocities can be written as

$$\frac{df}{d\zeta} = u_\xi - iu_\eta = \int_{-2R}^{+2R} -\frac{i\gamma(\nu)d\nu}{2\pi(\zeta-\nu)} \qquad \text{(Bgee6)}$$

Extracting the velocity components

$$u_\xi = U\cos\alpha - \frac{1}{2\pi}\int_{-2R}^{+2R}\gamma(\nu)d\nu\left\{\frac{\eta}{(\nu-\zeta)^2+\eta^2}\right\} \qquad \text{(Bgee7)}$$

$$u_\eta \; = \; U \sin\alpha + \frac{1}{2\pi}\int_{-2R}^{+2R} \gamma(\nu) d\nu \left\{ \frac{(\nu-\zeta)}{(\nu-\zeta)^2+\eta^2} \right\} \qquad \text{(Bgee8)}$$

where we have added the contributions from the uniform stream, U. The next step is to apply the condition of zero normal velocity on the surface of the foil, namely $u_\eta = 0$ for $\eta = 0$, $-2R < \xi < 2R$. Applying this to equation (Bgee8) generates an integral equation for the unknown distribution $\gamma(\xi)$. The technique for solving such an integral equation is beyond the scope of this text but it can be shown that, after the application of the Kutta condition (which eliminates another possible solution), this leads to

$$\gamma(\xi) \; = \; -2U \sin\alpha \left\{ \frac{2R-\xi}{2R+\xi} \right\}^{\frac{1}{2}} \qquad \text{(Bgee9)}$$

and this generates the previously documented solution. This demonstrates that it is possible to solve such simple planar potential flows directly using the methods of complex variables rather than relying on the indirect approach of discovering flows that correspond to particular stipulated functional forms of the complex potential.]