Vortex Sheets

A very useful concept and tool that can be synthesized using a potential vortex is the idea of a vortex sheet. To introduce this, consider the infinite linear distribution of free vortices of circulation, Γ, and spacing, a, shown in Figure 1 and the resulting planar flow streamlines produced by that arrangement of vortices.

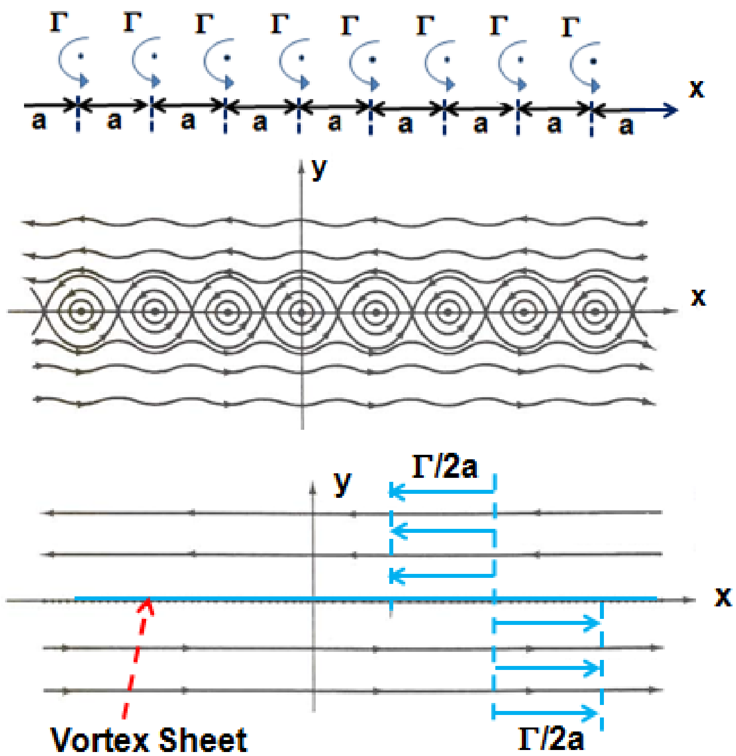

Figure 1: Infinite linear distribution of free vortices of circulation, Γ, and spacing, a, (top), the resulting streamlines (middle) and the vortex sheet flow as seen from afar (bottom).

Viewed from afar this planar flow field reduces to a velocity discontinuity with a near-uniform stream of velocity $\Gamma/2a$ in the negative x direction above the x axis and a near-uniform stream of velocity $\Gamma/2a$ in the positive x direction below the x axis as seen in Figure 1(bottom). This is termed a **vortex sheet** and is characterized by the magnitude of the velocity discontinuity, namely $U = \Gamma/a$ in this simple case. The magnitude of that velocity discontinuity, U, is called the **vortex sheet strength**, denoted by γ.

The concept and use of a vortex sheet can then be expanded by considering a planar potential flow with a solid boundary, a piece of which is sketched in Figure 2. The boundary is divided into a series of elements of length, ds, and we simulate the flow by considering that each element, ds, consists of a vortex sheet element of strength, $\gamma(s)$. Then the flow induced at any general point in the flow field, (x, y), by the vortex sheet element of strength, $\gamma(s)$, whose position is $(x_s(s), y_s(s))$ will be

$$\phi = \frac{\gamma(s)\ ds}{2\pi} \arctan\left\{\frac{(y - y_s(s))}{(x - x_s(s))}\right\} \quad ; \quad \psi = -\frac{\gamma(s)\ ds}{4\pi} \ln\left\{(x - x_s(s))^2 + (y - y_s(s))^2\right\} \qquad \text{(Bgdl1)}$$

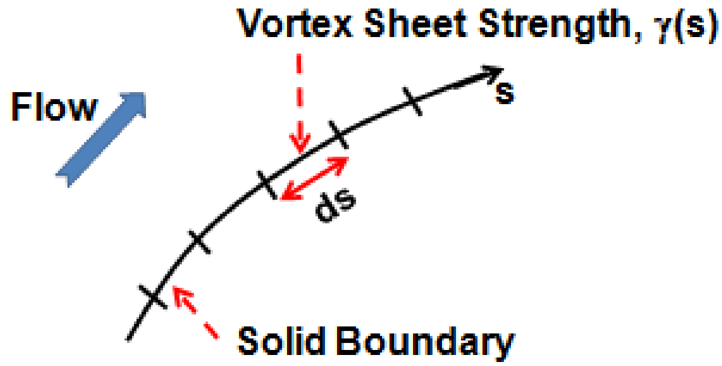

Figure 2: Section of the solid boundary of a planar potential flow.

according to equation (Bgdc3). Summing the contributions from all the elements of the surface, S, the flow field is then simulated by

$$\phi \;=\; \int_S \frac{\gamma(s)\;ds}{2\pi}\arctan\left\{\frac{(y-y_s(s))}{(x-x_s(s))}\right\} \quad ; \quad \psi \;=\; -\int_S \frac{\gamma(s)\;ds}{4\pi}\ln\left\{(x-x_s(s))^2+(y-y_s(s))^2\right\} \quad \text{(Bgdl2)}$$

plus contributions from other components such as a uniform stream. The velocity components then follow by differentiation of ϕ or ψ. Of course, the vortex element strengths, $\gamma(s)$, are, as yet, unknown and this determination is made by applying the boundary condition that the velocity induced normal to the surface of each element must be zero.

Numerically the method therefore proceeds as follows. The solid surface(s) is divided into J elements of strength, γ_j, $j = 1, ..., J$, length, s_j, and position $(x - x_j), (y - y_j)$. Then using

$$\phi \;=\; \sum_{j=1}^{J} \frac{\gamma_j\;s_j}{2\pi}\arctan\left\{\frac{(y-y_j)}{(x-x_j)}\right\} \quad ; \quad \psi \;=\; -\sum_{j=1}^{J} \frac{\gamma_j\;s_j}{4\pi}\ln\left\{(x-x_j)^2+(y-y_j)^2\right\} \quad \text{(Bgdl3)}$$

plus contributions from other components such as a uniform stream. Then the induced velocities at the center of each of the surface elements are determined (in doing so we omit from the summations any contribution from the vortex element where we are evaluating the velocity). This yields numerical expressions for the J normal velocities in the center of each of the elements. These must all be zero and this generates J equations which must then be solved to determine the J vortex sheet strengths, γ_j. This is the simplest form of what is known as the **boundary element method** based on vortex sheet elements. It has the advantage over the doublet distribution method in that the known shape of the object is inputted at the beginning rather than simply being discovered as the calculation proceeds.

The basic methodology described above was first applied to planar potential flows. Later the method was extended to axisymmetric flows with the development of the velocity potential due to a short cylindrical vortex element. Moreover, a fully three-dimensional version of this vortex sheet element method was used by Douglas Aircraft Company in the first successful attempts to compute the potential flow past a complete aircraft.